ALI

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Topology Workshop *1*

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The aim of this workshop is to provide you with the basic tools of Lacan's topology. Lacan's use of topology has spread, not only in the field of analysis—probably not among professional mathematicians, except perhaps a few in knot theory—but also in what the honest man of the 21st century needs to know about mathematics. In mathematics sections, particularly on the Internet, one will find analytical topology, in the mathematical sense, differential topology, and then one will also find Jacques Lacan's topology. This is because Lacan identified in topology a number of writings on which he could base his discourse—and not without being surprised, as we will have occasion to develop.

I Trivial topology

I would like to start with a topology that is called **trivial**, the simplest possible: the topology of *the whole* and the *void*.
The goal is to understand what a topology, a topological space, is. We have representations, we see surfaces, we may see knots, but for a mathematician these are imaginary objects. Topology is first and foremost a form of writing, and it is this form of writing that I would like to introduce to you.

The simplest set—since topology is a structure that applies to a set—is the set that contains nothing, $E = \{ \varnothing \}$, or the empty set.
Could I write it like this: $E = \{ \}$ since there is nothing in this set? I have used the terms *empty* and *nothing,* but this notation—these two ways I have written the set E—shows you that empty is not the same as nothing: empty is not nothing.

From this set, we can construct what is called the set of parts. What are the parts of a set in which there is nothing? This is an important question that mathematicians ask themselves.
Well, by definition, the set of parts of E is the set of all subsets formed from elements of E: that is, in this case, E itself and the empty set ∅. We write it as: $P(E) = \{ E, \varnothing \}$

And even on such a small set, the set of parts of the empty set, we can build a **topological space**. What is a topological space? It is a family: in the set of parts, we say that we have established a topology if we have distinguished a family of subsets that satisfy three properties.

First, it's the job of topology: to write. To understand—we'll get to that later. Here are the three properties:

1^{e}property: $E \in \sigma$ and $\varnothing \in \sigma$
The entire set, E, and the empty set, ∅, belong to this family (here called σ)

2^{e} property: $E \cup \varnothing \in \sigma$
If I combine several subsets of the family, I always remain within the family. That is, the union of the subsets is also in the family. This property must be verified for an infinite number of unions.

3^{e} property: $E \Omega \varnothing \in \sigma$
What is common to two subsets, the intersection, also belongs to the family.
This property must be verified for any finite number of intersections.

Three properties, and you can see that we are very far from what we can **imagine**: it is a form of writing. And this writing will allow us to construct topologies on spaces that we cannot imagine, or that we cannot conceive of. The work that Lacan did in relation to Freud was precisely to move away from the specular. Lacan moved away from the specular and moved on to a writing.

So why, in this first, very simple case, that of the set of parts of E, of $\sigma = \{ E, \varnothing \}$, do we indeed have a topology?
1^{e} property. First, because the first property is verified: each of the subsets, and in the trivial topology we take them all, that is, in this case E and ∅, each belongs to this set.
$E \in \sigma$ and $\varnothing \in \sigma$

2^{e} property. I take the union of E and the empty set, ∅, that is, everything that is in one or the other or both. The union corresponds to a kind of addition of sets (it differs from the usual addition but has the same function), and the empty set ∅ is its zero.
The union of E and ∅ is equal to E. $E \cup \varnothing = E$ Therefore,
$E \cup \varnothing \in \sigma$

3^{e} property. What E and the empty set ∅ have in common is the empty set. And both E and the empty set belong to the family.
$E \Omega \varnothing = \varnothing$, so we have $E \Omega \varnothing \in \sigma$

So, these properties having been verified, this does indeed constitute a topology of this minimal space, and I have called this family σ to remind us that in topology, these are called **open** sets. In other words, defining a topology on a set means defining which sets in that set are open.

- *Speaker: Why the term "open"?*

Because, after all, the imagination supports the thinking of mathematicians. And we'll see this later, when we get to more concrete things... Here, I've simply stated the skeleton, later we'll give a little flesh to all this—mathematical flesh—and we'll see the open...

All of this was really obvious, trivial, and one wonders why it should be written down. But let's now take a set in which there is something, in addition to the empty set, a set in which there are 3 elements, elements a, b, and c: E = { a, b, c }.
If I want to write a topology, using the trivial topology again, I will
list the set of parts of E, that is, the list of all subsets of E:
There is E itself, because the word part is not understood in the strict sense—as being strictly inside—but rather as being inside or being the same thing. Then, we can write down all these subsets: the one formed of 3 elements, i.e., E, those formed of 2 elements, those with only 1 element, and the one with no elements: the empty set.
$P_{(E)}$ = { E, {a,b} , {b,c}, {c,a}, {a}, {b}, {c}, ∅ }
So here we have the set of all parts of E.
How does this verify the topology? Does taking all these subsets constitute a family of open sets, i.e., a family of subsets that verifies the three properties? Let's take a look:

1e property: E and the empty set ∅ must be in the family. Let's check: they are.

2e property: the union of two parts. There are many possible unions of two parts in this family: for example, the union of a and b gives the set {a, b}, which is part of the family
{a} ∪ {b} = {a,b} ∈ $P_{(E)}$
The same applies to a and c and b and c.
This is therefore verified for unique elements. Is it verified, for example, for {a,b} ∪ {c}?
{a,b} ∪ {c} ∈ $P_{(E)}$, since {a,b} ∪ {c} = { a, b, c } and { a, b, c } = E
If I take {a,b} ∪ {a}, we have {a,b} ∪ {a} = {a,b}, and {a,b} ∈ $P_{(E)}$.
We check all possible cases, and this second property is verified.

3e property, for the intersection. If I take, for example, the intersection of {a,b} with a: {a,b} Ω {a}= {a} and {a} ∈ $P_{(E)}$.
The same applies if I take the intersection of {a,b} and {c }: {a,b} Ω {c} = ∅ and ∅ ∈ $P_{(E)}$.

This third property is therefore also verified.

The family of subsets, of all possible subsets of the set, does indeed constitute a topology. And it is a topology that is said to **be trivial** since it involves all subsets.

II Non-trivial topology

What is a **non-trivial** topology? I will continue with the same example, E = ⎨ a, b, c ⎬, and we will construct another topology on this same set.
We will consider the family σ, in which we will put E and ∅—we have to, it's the first condition—and we will add only three elements: ⎨a⎬, ⎨b⎬, and ⎨a,b⎬.
σ = ⎨ E, ∅, ⎨a⎬, ⎨b⎬, ⎨a,b⎬ ⎬
Does this family σ satisfy our three properties?

1e Property: E and the empty set ∅ must be elements of the family. They are.

2e property: the union of two parts. For example, ⎨a⎬ ∪ ⎨b⎬ = ⎨a,b⎬ and ⎨a,b⎬ ∈ σ.
Let's take ⎨a⎬ ∪ ⎨a,b⎬ = ⎨a,b⎬, which is still part of σ. You can try others, but in all cases we obtain an element that belongs to the family.
Therefore, the second property is true.

3e Property, for the intersection. ⎨a,b⎬ Ω ⎨a= ⎨a⎬ and ⎨a⎬ ∈ σ
Or ⎨a⎬ Ω ⎨b⎬ = ∅ and ∅ ∈ σ. Still elements of the family. The third property is verified.

This is more interesting because not all subsets of E are in this family of open sets σ: for example, ⎨c⎬, ⎨b,c⎬, and ⎨c,a⎬ are not in it. ⎨c⎬ is present in E, but it is excluded from all intersections and can only appear accompanied by ⎨a⎬ and ⎨b⎬.

In any case, since the three properties are verified, we have indeed defined a family of open sets, and it is this family of open sets that gives E a topology.

- *I have one objection*: *it is not E that is given a topology, but a subset of E.*

It gives a topology to E—which is as given: E = ⎨ a, b, c ⎬—because I add something to E that is not given at the outset, that is not already in E. I add a family of open sets that I write like this:

$\sigma = \{ E, \varnothing, \{a\}, \{b\}, \{c\} \}$.

- *But it's not an addition, it's a partition of a subset of E!*

Yes, absolutely. I choose a subset of E. And it is the act of choosing a subset of E that gives E its topology—which I chose by choosing this subset of E.

- *There is a shift: you attribute to E the property of a subset.*

We agree that σ is not a complete partition of E, but a partition of a subset of E. And I chose this subset of E without including {c}. I could have chosen not to include {b}: it's a choice. Do you agree that I can make this choice? Good. So, having made this choice, I write a partition of this subset. In fact, I don't even have to write a partition of this subset: I chose to make a partition of a subset, that is, to write a family of subsets.
The only important point is not the fact that it is a partition of a subset, but the fact that it is a family of subsets.

You still don't seem to agree? So, there is indeed a shift for you, and I will try to respond.
The reasoning I used for σ, with the three properties, I used for the family of open sets. Do you agree?

- *Yes.*

Whether this family of open sets is a partition of a subset is not the essential point here: what is important is that it is a family. Furthermore, I have carefully checked that the three properties are verified.
So, to a set of E, I associated a family of subsets satisfying the
three properties. That is, within E there is a family, and within this family the three properties are true. And this confers a topology on E. That is the definition. I suggest you accept this as a definition or look it up in Bourbaki: the family gives E a topology. And by virtue of the first property, E is part of the family.

- *Virginia Hasenbalg: When you're not a mathematician, it's a way of clearly seeing the concept of internal exclusion. We're going to perform operations with E—E being included in the subsets, E being in E, without it being written—and that allows us to identify something of an internal exclusion: we're also in a Lacanian topology.*

Here, we are also in a mathematical topology.

- *Speaker: You didn't say it, but it's probably a difference between class logic and set logic: every set includes the empty set... When in "From One Other to the Other" Lacan calls the flaw in the Other*
"the empty set" and not "zero," and he takes care to explain why.

This question of the empty set, for us in analytical discourse, has to do with the castration of the Other, the flaw in language. So why does every set necessarily include the empty set?

What is a subset? A subset is a set that contains 0, 1, several, or all the elements of another set. This gives us all the possible subsets—this is what we call a partition. And in this list, there is: "take no elements." A subset may contain no elements: this is the empty set.
To expand on your question: why must the empty set, ∅, be part of the open sets? Well, because we sought to construct an object, a topology, that
could be usable: if we didn't include the empty set, the third property couldn't be verified, or else there would only be one element.

- *In "From One Other to the Other," Lacan says that he does not use 0, because to use it, one would have to be able to use numbers—the implication being that we do not know how to count beyond 2. He therefore uses the empty set as he uses the One as a signifier and not as a countable 1. For example, we can think that in the One that supports identification, this One is a signifier.*
Could the inclusion of the empty set in a set take the place of SA?

To answer briefly: Lacan at this point follows Frege's approach, and so we will count the writings of empty sets.

I will continue and we will come back to these questions later: today I would like to be brief and concise.

One thing to remember is that on the same set E, we have defined two topologies: a first topology where the complete partition of E was constituted as open, and a second topology constituted here by only certain subsets of E.
If there is only one thing to remember here, it is that topology is not induced by the set—it is not in the set. It is something that we add: it is an interpretation of the set that will give it a topology. And so the same set can be the support for several topologies; this is important since proximities—topology will then involve notions such as *that of* neighborhood—will not be the same. For example, if it were a set of points, it is not only the metric distance that will determine whether one point is close to another.

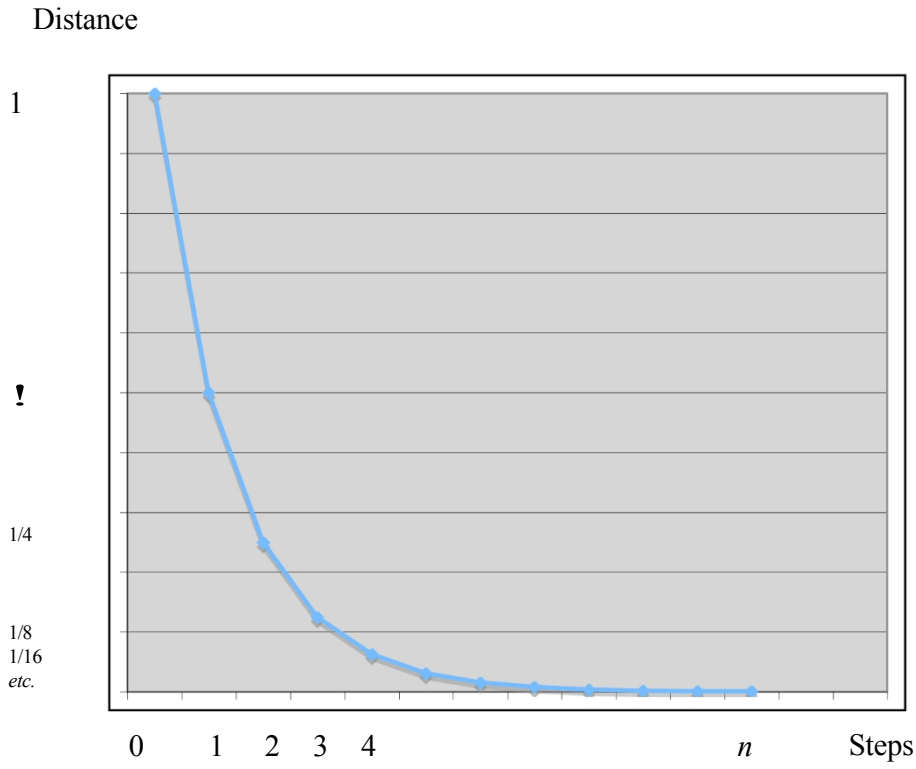
The two topologies I have just proposed are therefore, first, the trivial topology—where all subsets were considered, all open—and second, the non-trivial topology.
That said, it is still a topology built on a finite set: and a topology built on a finite set does not go far, it is limited. Where it gets interesting is when there is infinity.

- *You told us that topology was a form of writing. And now you're adding reading to that...*

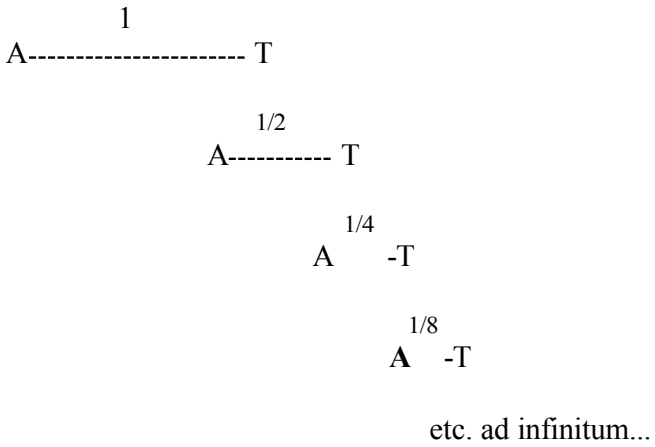
A reading, yes. We have produced several writings on a set: trivial writing and non-trivial writing, which give rise to several readings.

III The question of infinity.

To move forward with the question of infinity, we will discuss the paradox of Achilles and the tortoise. You can imagine that this is not a paradox at all: because if Achilles is faster than the tortoise, there will come a point when he catches up with it and overtakes it. What is paradoxical is that Zeno developed it as a paradox.
How would we write it today if we followed Zeno's reasoning? Well, there are a number of steps, represented on the horizontal axis, and at each step Achilles moves forward one step. And we plot on the vertical axis the distance he has left to cover to catch up with the tortoise: at zero, he has the whole distance to cover, then he moves forward and each time he covers half the distance to catch up with the tortoise. So at the beginning, there is 1, i.e., the entire distance, then ! at the next step, then ", etc. So you have a distance that decreases along this curve—without the blue line ever exactly coinciding with the axis (*which is difficult to represent on paper)*.

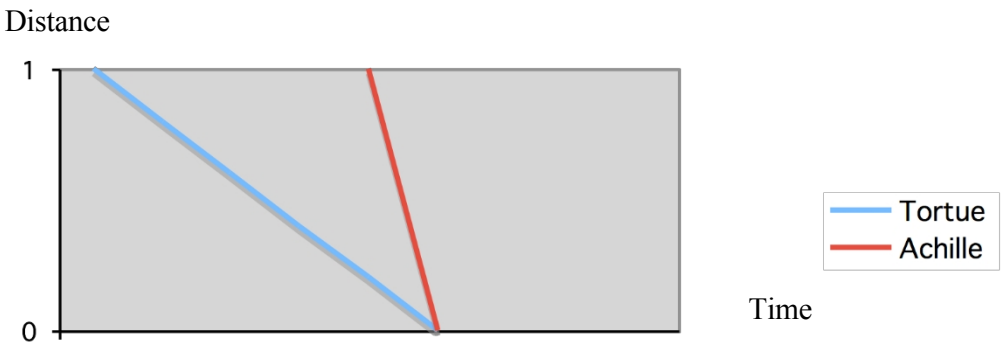


And Zeno's reasoning is that Achilles will never reach the point where he should meet the tortoise, since at each stage he only covers half the distance between them.



The principle is that there is always something between Achilles and the tortoise: if there is 1 millimeter left, then he will run 1 millimeter, and then " millimeters... In calculations, we can go very far, but there will always be what is called in mathematics an *epsilon:* a non-zero quantity as small as we want.
But then, why did this line of reasoning appear to be a paradox? Where does it go wrong? And why does Lacan refer to it? We see this, for example, in the first lesson of "*Encore":* reasoning in this way does not allow Achilles to catch up with the tortoise, and this is Zeno's argument.

Now, let's draw it today—as we could have done at the time, with two axes. On the horizontal axis, I indicate time rather than stages, and on the vertical axis, the distance remaining to be covered: the tortoise moves forward, the distance remaining decreases, and it will arrive at its destination. Achilles, on the other hand, sets off a little later, but he moves faster: so they will meet...



Why is there a paradox?

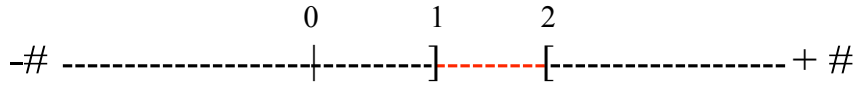
Here is my interpretation: in the first graph, with the single blue curve, we are dealing with something that is in the realm of the signifier, meaning that we are moving forward step by step, and at each step, I repeat the same sentence
" I am going to cover half of the distance that remains to be covered." It is the same sentence, and it always takes the same amount of time to say: and it does not take into account one thing, which is that these steps, as I get closer—if I am going at a constant speed—are getting smaller and smaller. So each step will take less and less time: they will take half as long each time. In other words, the points written on the horizontal axis of this diagram 1, points 1, 2, 3, 10, n, etc., have been evenly spaced so that I have time to articulate my sentence each time: "I will travel half the distance that remains to be traveled." This is something that is clearly articulated in what is said, and not in the physical representation of the phenomenon, as illustrated above by the diagram with the two lines, blue and red.
In other words: it is not in reality that there is a paradox in the encounter between Achilles and the tortoise—it is in the saying. Which is no small thing... especially when the encounter can only exist in the saying.
So, what I did based on this approach was to introduce the notion of infinity and see that as soon as infinity is mentioned, new questions arise, and to address these questions correctly—that is, not imaginatively—the support of topology will be very useful.

So here, in this story about the turtle, at the level of infinity, we have imaginary effects since we are dealing with an infinitely small distance that is traveled in an infinitely small amount of time. What does that give us? It turns out that it gives us something finite, but we can multiply infinity by infinity—the infinitely large by the infinitely small—and what is the result? Well, these are things that call upon notions of infinity, which we need to articulate with writing.

IV A topology of infinity.

A few words about what a topology of infinity might be.
The simplest topology of infinity would be that of a straight line: a distance that goes from something very small to something very large, from something very far away on one side to something very far away on the other side.

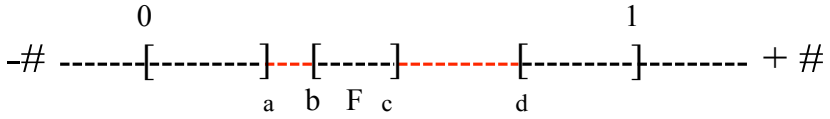
a Open interval
On this line, the most common topology—and so this is indeed an additional definition: it is something that is not on the line, they are simply numbers that are classified—the most common topology is that of constituting pieces of this line that we write like this,] in square brackets [because we exclude the ends.



For example, it is all the points between 1 and 2—except 1 and 2. And so it is the same as Achilles and the tortoise: I can get as close as I can to 1, and find myself in this interval marked in red, always in this interval—never on 1. I will always still have half of some distance, some epsilon, to travel to get there: I will always be in this interval. This interval is called **open** because the endpoints do not belong to this interval.
So you see that we have infinity at each end of the line, but we also have it inside the line.

β Closed interval
We are now going to work on a part of this line, on all the points between 0 and 1, and you can see that here I am putting the intervals with the brackets on the inside: this is what we call a **closed** interval**.**

γ Open and closed intervals
And within this closed segment [0,1], we will consider a set of open segments, i.e., segments that do not have endpoints: these are the ones shown in red on the right below:]a,b[and]c,d[. They can go as close as possible to the endpoints, but without ever touching them.



In topology, a **closed** set is the complement of an **open set**. And of course we have the opposite: the complement of an open set is a closed set.
On the right, the closed set F, i.e., [b,c], has as its complement the open set O, i.e.,]a,d[, represented in red on the right, and O is the complement of F.
And the three properties we wrote down earlier can easily be written using *closed* sets.

Incidentally, note that the complement of the set E is the empty set ∅. And that the complement of the empty set ∅ is the set E.
So E and ∅ are both open and closed.
Because, in E, the complement of E is the empty set ∅. The complement of the empty set ∅ in E is E.

If E is open—and E is open, as we have seen—its complement is *closed,* so the empty set ∅ is closed. This is a direct consequence of the notation: you have sets that are both open and closed.

So the empty set is both open and closed, and in the same way, E is both open and closed.

All these open sets, you can see that there are infinitely many between 0 and 1 since there are infinitely many ways to place the left edge], and infinitely many ways to place the right edge [. So a double infinity, which is still an infinity of the same order...

- *Infinity on the right edge, infinity on the left edge? I don't understand...*

The open interval]a,d[, which is shown in red in the diagram above between [0 and 1], could have started at any point: in fact, there are an infinite number of places where I could have started it. And there are an infinite number of places where I could have ended it... Because between 0 and 1, how many points are there?

- *An infinite number.*
- *This brings us back to Achilles!*

Yes, between [0 and 1] there is 1/2, there is 1/4, 1/8, etc. I can even write that there is 1/n, and since n can go up to #, I will have an infinite number of points as I approach 0. And the same reasoning applies when I approach 1, with an infinite number of points. I can apply this reasoning at any point on this segment: so I really do have an infinite number of points.
This infinity is even more infinite than the number *n*: this is what Cantor established with his diagonal. In other words, the infinity of points between 0 and 1 is denser, more significant, of a different order, than the infinity that goes from 0 to # when we count 1, 2, 3, etc. It is an infinity that is denser than the accountant's infinity.
These open spaces can be in an infinite number of positions on this line, and we can have an infinite number of them.

Why are we talking about this? Because Lacan will deduce from it, will use it to write about the impossibility of sexual intercourse.
For now, let's stick with the fact that between [0 and 1] we can have an infinity of open spaces—which may overlap partially, etc.
We also have an infinite number of closed spaces; we could construct the same thing using the complements.

- *Does Lacan talk about the impossibility of sexual relations based on both closed and open spaces?*

He takes this from the open spaces since the topology we have constructed is based on the open spaces. We could do the same thing starting from the complements.

But the open sets are perhaps more interesting: we will see that this is the feminine side.

- *In both cases, there is no relationship? There is no point where the two meet? There is a limit in the closed ones...*

There is a limit in closed sets, but this limit will be a point. Is this point part of the closed set family? At this point, we would also have to consider that our list contains closed sets that are reduced to one of the points, but this is not the approach we have taken since we have been working on open sets.

Let's leave it there for today...

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